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UNSTEADY HEAT TRANSFER IN A SYSTEM OF THREE COAXIAL FINITE CYLINDERS

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An integral transformation with the aid of which a solution of the problem of unsteady-state heat transfer in a system of three coaxial finite cylinders with different boundary conditions on their surfaces depending on space and time is presented. Each of the cylinders evolves heat of a certain intensity, depending on time and coordinates. A numerical solution of one variant of the boundary conditions is given and illustrated by figures. The method of transforming the solution of the problem with other boundary conditions is shown.

Investigations are known [1–3] pertaining to problems of heat transfer in conjugated bodies of different shapes with substantial restrictions in the boundary functions in which the Laplace and Hankel integral transformations are used. In [4], a solution of the problem of heat conduction for two conjugate bodies, which is based on the author's theory of Fourier–Hankel integral transformations, is given. However, it is indicated in that work that the approach developed can be used only in certain simple cases, viz., at constant values of the boundary functions in the problem statement.

In the present work and in [5], an integral transformation is suggested, which differs from the well-known ones and which allows one to obtain the solution of the heat-conduction problem for two, three, or more conjugate bodies with boundary functions depending on space and time. This makes it possible to considerably expand the class of the heat-conduction problems solvable analytically. It should be noted that the proposed form of solution can be used with slight changes for the problems of a body of a given shape with all kinds of boundary functions.

Below, as an example, a nonstationary axisymmetric problem of heat conduction is considered in a system of three coaxial finite cylinders, on the outer boundaries of which boundary functions can be adopted, which, in what follows, are called the boundary conditions of heat transfer of the first, second, or third kind depending on both space and time. At the boundary of conjugation of the cylinder layers there is a complete thermal contact (boundary conditions of the fourth kind). Moreover, it is assumed that the cylinder layers generate heat, the release of which depends also on space and time.

To obtain a solution of the problem, finite integral transformations with respect to two coordinates of the cylinder are used. In the solution presented, one of the possible variants of boundary conditions is considered, but, as will be shown below, it can be easily altered to obtain a solution with other boundary conditions.

The cylindrical system of coordinates and the geometry of the cylinder are presented in Fig. 1. The problem of determining the temperature field in the cylinder can be presented in the form of heat-conduction equations and conditions on the outer boundaries and on the boundaries of conjugation of the cylinder layers, as well as initial conditions. The following boundary conditions are adopted: on the inner cylindrical surface $r = R_1$ — the condition of the 2nd kind; on the outer cylindrical surface $r = R_4$ — the condition of the 3rd kind; on the lower end of the cylinder — the condition of the 2nd kind, and on the upper end — the condition of the 3rd kind.

Mathematically, the problem posed can be represented by the heat-conduction equations:

$$\frac{1}{a_i}\frac{\partial T_i}{\partial \tau} = \frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r} + \frac{\partial^2 T_i}{\partial z^2} + \frac{1}{\lambda_i} w_i \,, \quad T_i = T_i(\tau, r, z)\,, \quad R_i < r < R_{i+1}\,, \quad b < z < c\,, \quad i = 1, 2, 3\,; \tag{1}$$

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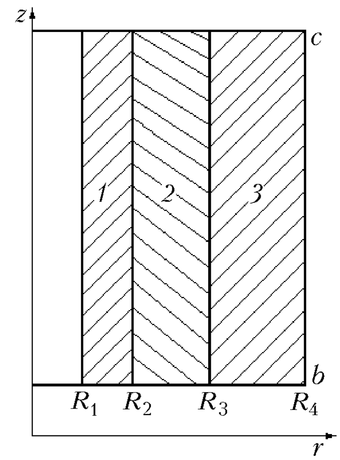


Fig. 1. Geometry of the cylinder and the coordinate system: 1–3, numbers of cylinders.

by boundary conditions on the inner cylindrical surface

$$\lambda_1 \frac{\partial T_1}{\partial r}\bigg|_{r=R_1} + q(\tau, z) = 0\,, \tag{2}$$

on the outer cylindrical surface

$$\lambda_3 \frac{\partial T_3}{\partial r}\bigg|_{r=R_4} + \alpha_1 \left[T_3 \big|_{r=R_4} - T^{\mathrm{m}}(\tau, z) \right] = 0\,, \tag{3}$$

on the lower and upper ends of the cylinder, respectively,

$$\lambda_i \frac{\partial T_i}{\partial z}\bigg|_{z=b} + q_{i+1}(\tau, r) = 0\,, \quad R_i \le r \le R_{i+1}\,, \quad i = 1, 2, 3\,, \tag{4}$$

$$\lambda_i \frac{\partial T_i}{\partial z}\bigg|_{z=c} + \alpha_{i+1} \left[T_i \big|_{z=c} - T^{\mathrm{m}}_{i+1}(\tau, r) \right] = 0\,, \quad R_i \le r \le R_{i+1}\,, \quad i = 1, 2, 3\,; \tag{5}$$

on the boundaries of conjugation of the cylinder layers

$$T_1 \big|_{r=R_2} = T_2 \big|_{r=R_2}\,, \quad T_2 \big|_{r=R_3} = T_3 \big|_{r=R_3}\,, \quad \lambda_1 \frac{\partial T_1}{\partial r}\bigg|_{r=R_2} = \lambda_2 \frac{\partial T_2}{\partial r}\bigg|_{r=R_2}\,, \quad \lambda_2 \frac{\partial T_2}{\partial r}\bigg|_{r=R_3} = \lambda_3 \frac{\partial T_3}{\partial r}\bigg|_{r=R_3}\,; \tag{6}$$

and by the initial conditions

$$T_i \big|_{\tau=0} = T_{i0}(r, z)\,, \quad i = 1, 2, 3\,. \tag{7}$$

Here, 1, 2, and 3 denote the inner, middle, and outer cylinders (see Fig. 1).

We will introduce the following change of variables:

$$z = \sqrt{a_i}\, y\,, \quad r = \sqrt{a_i}\, x \quad \text{when} \quad b \le z \le c\,, \quad R_i \le r \le R_{i+1}\,, \quad i = 1, 2, 3\,. \tag{8}$$

Then Eqs. (1)–(7) will take the form

$$\frac{\partial Z_i}{\partial \tau} = \frac{\partial^2 Z_i}{\partial x^2} + \frac{1}{x}\frac{\partial Z_i}{\partial x} + \frac{\partial^2 Z_i}{\partial y^2} + \frac{a_i}{\lambda_i} w_{ii} ,$$

$$Z_i (\tau, x, y) = T_i (\tau, r, z) , \quad x_{2i-1} < x < x_{2i} , \quad y_{2i-1} < y < y_{2i} , \quad i = 1, 2, 3 ; \tag{9}$$

$$b_1 \frac{\partial Z_1}{\partial x}\bigg|_{x=x_1} + qu (\tau, y) = 0 , \quad y_1 \le y \le y_2 ; \tag{10}$$

$$b_3 \frac{\partial Z_3}{\partial x}\bigg|_{x=x_6} + \alpha_1 \left[Z_3 |_{x=x_6} - \theta^{\mathrm{m}} (\tau, y) \right] = 0 , \quad y_5 \le y \le y_6 ; \tag{11}$$

$$b_i \frac{\partial Z_i}{\partial y}\bigg|_{y=y_{2i-1}} + qu_{i+1} (\tau, x) = 0 , \quad x_{2i-1} \le x \le x_{2i} , \quad i = 1, 2, 3 ; \tag{12}$$

$$b_i \frac{\partial Z_i}{\partial y}\bigg|_{y=y_{2i}} + \alpha_{i+1} \left[Z_i |_{y=y_{2i}} - \theta^{\mathrm{m}}_{i+1} (\tau, x) \right] = 0 , \quad x_{2i-1} \le x \le x_{2i} , \quad i = 1, 2, 3 ; \tag{13}$$

$$Z_1 |_{x=x_2} = Z_2 |_{x=x_3} , \quad Z_2 |_{x=x_4} = Z_2 |_{x=x_5} , \quad b_1 \frac{\partial Z_1}{\partial x}\bigg|_{x=x_2} = b_2 \frac{\partial Z_2}{\partial x}\bigg|_{x=x_3} , \quad b_2 \frac{\partial Z_2}{\partial x}\bigg|_{x=x_4} = b_3 \frac{\partial Z_3}{\partial x}\bigg|_{x=x_3} ; \tag{14}$$

$$Z_i |_{\tau=0} = \theta_{i0} (x, y) , \quad i = 1, 2, 3 , \tag{15}$$

where

$$b_i = \lambda_i / \sqrt{a_i} ; \quad w_{ii} (\tau, x, y) = w_i (\tau, r, z) ; \quad q_{i+1} (\tau, r) = qu_{i+1} (\tau, x) ; \quad T^{\mathrm{m}}_{i+1} (\tau, r) = \theta^{\mathrm{m}}_{i+1} (\tau, x) , \quad i = 1, 2, 3 ;$$

$$q (\tau, z) = qu (\tau, y) ; \quad T^{\mathrm{m}} (\tau, z) = \theta^{\mathrm{m}} (\tau, y) ;$$

$$x_1 = R_1/\sqrt{a_1} ; \quad x_2 = R_2/\sqrt{a_1} ; \quad x_3 = R_2/\sqrt{a_2} ; \quad x_4 = R_3/\sqrt{a_2} ; \quad x_5 = R_3/\sqrt{a_3} ; \quad x_6 = R_4/\sqrt{a_3} ;$$

$$y_1 = b/\sqrt{a_1} ; \quad y_2 = c/\sqrt{a_1} ; \quad y_3 = b/\sqrt{a_2} ; \quad y_4 = c/\sqrt{a_2} ; \quad y_5 = b/\sqrt{a_3} ; \quad y_6 = c/\sqrt{a_3} . \tag{16}$$

To solve system (9)–(15), the following integral transformation will be determined:

$$\overline{Z} (\tau, s, y) = \sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} x Z_i (\tau, x, y) \, U_i (sx) \, dx , \tag{17}$$

where s denotes the roots of the characteristic equation presented below; $U_i(sx)$ satisfies the following equations and boundary conditions:

$$\frac{d^2U_i}{dx^2} + \frac{1}{x}\frac{dU_i}{dx} + s^2U_i = 0 , \quad x_{2i-1} < x < x_{2i} , \quad i = 1, 2, 3 ; \tag{18}$$

$$\left.\frac{dU_1}{dx}\right|_{x_1} = 0 , \quad U_1\,|_{x_2} = U_2\,|_{x_3} , \quad b_1\left.\frac{dU_1}{dx}\right|_{x_2} = b_2\left.\frac{dU_3}{dx}\right|_{x_3} , \quad U_2\,|_{x_4} = U_3\,|_{x_5} ,$$

$$b_2\left.\frac{dU_2}{dx}\right|_{x_4} = b_3\left.\frac{dU_3}{dx}\right|_{x_5} , \quad b_3\left.\frac{dU_3}{dx}\right|_{x_6} + \alpha_1 U_3\,|_{x_6} = 0 . \tag{19}$$

Equations (18) have the solutions [6]

$$U_i\,(sx) = C_{2i-1}\,J_0\,(sx) + C_{2i}Y_0\,(sx) , \quad i = 1, 2, 3 ,$$

where C_i are arbitrary constants; $J_0(sx)$ and $Y_0(sx)$ are the Bessel functions of the 1st and 2nd kinds and order zero. Conditions (19) yield the following system of equations:

$$\begin{gathered}
C_1J_1\,(sx_1) + C_2Y_1\,(sx_1) = 0 , \\
C_1J_0\,(sx_2) + C_2Y_0\,(sx_2) - C_3J_0\,(sx_3) - C_4Y_0\,(sx_3) = 0 , \\
b_1\,(C_1J_1\,(sx_2) + C_2Y_1\,(sx_2)) - b_2\,(C_3J_1\,(sx_3) + C_4Y_1\,(sx_3)) = 0 , \\
C_3J_0\,(sx_4) + C_4Y_0\,(sx_4) - C_5J_0\,(sx_5) - C_6Y_0\,(sx_5) = 0 , \\
b_2\,(C_3J_1\,(sx_4) + C_4Y_1\,(sx_4)) - b_3\,(C_5J_1\,(sx_5) + C_6Y_1\,(sx_5)) = 0 , \\
- b_3s\,(C_5J_1\,(sx_6) + C_6Y_1\,(sx_6)) + \alpha_1\,(C_5J_0\,(sx_6) + C_6Y_0\,(sx_6)) = 0 ,
\end{gathered} \tag{20}$$

which has a nontrivial solution only in the case where its determinant vanishes. This condition yields the characteristic equation for determining the numbers

$$\begin{vmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{vmatrix} = 0 , \tag{21}$$

where a_{11}, a_{12}, a_{21}, ... are the coefficients at C_1, C_2, C_3, ..., following from (20).

By applying simple transformations, having excluded interdeterminate constants C_2, C_3, C_4, C_5, and C_6, the following expressions will be obtained for the functions $U_1(sx)$, $U_2(sx)$, and $U_3(sx)$:

$$\begin{gathered}
U_1\,(sx) = C_1\,(J_0\,(sx) + f_1Y_0\,(sx)) , \\
U_2\,(sx) = C_1\,(f_2J_0\,(sx) + f_3Y_0\,(sx)) , \ U_3\,(sx) = C_1\,(f_4J_0\,(sx) + f_3Y_0\,(sx)) .
\end{gathered} \tag{22}$$

The coefficients f_i (i = 1, 2, 3, 4, 5) involve the quantities a_{11}, a_{12}, a_{21}, ...

In integral transformation (17) and expressions (22) the constants A_1, A_2, A_3, and C_1 remain interdeterminate. The values of A_1, A_2, and A_3 are determined from the condition of orthogonality of the functions $U_i(sx)$, which at $s \neq p$ presupposes the fulfillment of the relationships

$$\int_{x_1}^{x_2} xU_1(sx)\,U_1(px)\,dx = 0\,,\quad \int_{x_3}^{x_4} xU_2(sx)\,U_2(px)\,dx = 0\,,\quad \int_{x_5}^{x_6} xU_3(sx)\,U_3(px)\,dx = 0\,,$$

which follow from the equality

$$(s^2 - p^2)\left[A_1 \int_{x_1}^{x_2} xU_1(sx)\,U_1(px)\,dx + A_2 \int_{x_3}^{x_4} xU_2(sx)\,U_2(px)\,dx + A_3 \int_{x_5}^{x_6} xU_3(sx)\,U_3(px)\,dx\right] = 0$$

and boundary conditions (19) at $s \neq p$ and $A_1 = x_3b_1/x_2b_2$, $A_2 = 1$, and $A_3 = x_4b_3/x_5b_2$. The constant C_1 is determined from the condition of orthonormability of the functions $U_i(sx)$:

$$A_1 \int_{x_1}^{x_2} x\,[U_1(sx)]^2\,dx + A_2 \int_{x_3}^{x_4} x\,[U_2(sx)]^2\,dx + A_3 \int_{x_5}^{x_6} x\,[U_3(sx)]^2\,dx = 1\,.$$

Thus, the integral transformation (17) has been determined. After integral transformation, Eq. (9) yields

$$\sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} x\left(\frac{\partial Z_i}{\partial \tau} - \frac{\partial^2 Z_i}{\partial x^2} - \frac{1}{x}\frac{\partial Z_i}{\partial x} - \frac{\partial^2 Z_i}{\partial y^2} - \frac{a_i}{\lambda_i} w_{ii}\right) U_i(sx)\,dx = 0\,.$$

As a result of the transformation, the last relationship will take the form

$$\frac{\partial \overline{Z}}{\partial \tau} = \frac{\partial^2 \overline{Z}}{\partial y^2} + s^2 \overline{Z} + FP_1(\tau, s, y) + FP_2(\tau, s, y)\,, \tag{23}$$

$$FP_1(\tau, s, y) = A_1 x_1 U_1\big|_{x_1} \frac{qu(\tau, y)}{\lambda_1} + A_3 x_6 U_3\big|_{x_6} \frac{\alpha_1}{\lambda_3} \theta^{\mathrm{m}}(\tau, y)\,; \tag{24}$$

where

$$FP_2(\tau, s, y) = \sum_{i=1}^{3} A_i \frac{a_i}{\lambda_i} \int_{x_{2i-1}}^{x_{2i}} xw_{ii}(\tau, x)\,U_i(sx)\,dx\,. \tag{25}$$

By virtue of the determination of the integral transformation (17), Eq. (23) is valid within the ranges $x_1 \le x \le x_2$, $x_3 \le x \le x_4$, and $x_5 \le x \le x_6$. Its solution will be found within each of these intervals on having represented conditions (12) and (13) at the boundaries y_1, y_2, y_3, y_4, y_5, and y_6 in the form

$$b_i \frac{\partial \overline{Z}_i}{\partial y}\bigg|_{y=y_{2i-1}} + \overline{qu}_{i+1}(\tau, s) = 0\,,\quad b_i \frac{\partial \overline{Z}_i}{\partial y}\bigg|_{y=y_{2i}} + \alpha_{i+1}\left[\overline{Z}_i\big|_{y=y_{2i}} - \overline{\theta}^{\mathrm{m}}_{i+1}(\tau, s)\right] = 0\,,\quad i = 1, 2, 3\,, \tag{26}$$

here

$$\overline{qu}_{i+1}(\tau, s) = \sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} xU_i(sx)\,qu_{i+1}(\tau, x)\,dx\,;$$

$$\overline{\theta}_{i+1}^{m}(\tau, s) = \sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} xU_i(sx)\, \theta_{i+1}^{m}(\tau, x)\, dx\,, \quad i = 1, 2, 3\,.$$

To solve Eq. (23) within the ranges $y_1 \le y \le y_2$, $y_3 \le y \le y_4$, and $y_5 \le y \le y_6$, the following integral transformations will be determined:

$$\overline{\overline{Z}}_j(\tau, s, p) = \int_{y_{2j-1}}^{y_{2j}} \overline{Z}(\tau, s, y)\, V_j(py)\, dy\,, \quad j = 1, 2, 3\,, \tag{27}$$

where p stands for the roots of the characteristic equation presented below; $V_j(py)$ represents the solutions of the differential equations

$$\frac{d^2 V_j}{dy^2} + p^2 V_j = 0\,, \quad j = 1, 2, 3\,, \tag{28}$$

with boundary conditions following from (26):

$$\left.\frac{dV_j}{dy}\right|_{y_{2j-1}} = 0\,, \quad \lambda_j \left.\frac{dV_j}{dy}\right|_{y_{2j}} - \alpha_{j+1} V_j \big|_{y_{2j}} = 0\,, \quad j = 1, 2, 3\,. \tag{29}$$

The solutions of Eqs. (28) have the form

$$V_j(py) = D_{2j-1} \sin(p_j y) + D_{2j} \cos(p_j y)\,, \quad j = 1, 2, 3\,, \tag{30}$$

where D_1–D_6 are arbitrary constants. Subject to boundary conditions (29), the following system of equations is obtained:

$$D_{2j-1} \cos(p_j y_{2j-1}) - D_{2j} \sin(p_j y_{2j-1}) = 0\,,$$

$$D_{2j-1} \left[\lambda_j p_j \cos(p_j y_{2j}) + \alpha_j \sin(p_j y_{2j})\right] + D_{2j} \left[-\lambda_j p_j \sin(p_j y_{2j}) + \alpha_j \cos(p_j y_{2j})\right] = 0\,, \tag{31}$$

which has a nontrivial solution only in the case where its determinant vanishes. From this, the characteristic equation for determining p_j can be obtained:

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0\,, \tag{32}$$

where

$$b_{11} = \cos(p_j y_{2j-1})\,; \quad b_{12} = -\sin(p_j y_{2j-1})\,; \quad b_{21} = \lambda_j p_j \cos(p_j y_{2j}) + \alpha_j \sin(p_j y_{2j})\,;$$

$$b_{22} = -\lambda_j p_j \sin(p_j y_{2j}) + \alpha_j \cos(p_j y_{2j})\,, \quad j = 1, 2, 3\,.$$

Having excluded the coefficient D_{2j} from system (31), the expression for $V_j(p_j x)$ can be given in the form

$$V_j(p_j y) = D_{2j-1} \left[\sin(p_j y) + g_j \cos(p_j y)\right], \quad j = 1, 2, 3\,, \tag{33}$$

where

$$g_j = -\sin(p_j y_{2j-1})/\cos(p_j y_{2j-1}), \quad j = 1, 2, 3.$$

It should be noted that solutions (33) with boundary conditions (29) represent the orthogonal systems of the functions $V_j(p_j y)$. By requiring that this system of functions be orthonormalized, i.e.,

$$\int_{y_{2j-1}}^{y_{2j}} \left[V_j(p_j y)\right]^2 dy = 1,$$

the constants D_{2j-1} can be found. Thus, integral transformations (27) have been determined. Having applied the obtained integral transformations to relation (23)

$$\int_{y_{2j-1}}^{y_{2j}} \left[\frac{\partial \bar{Z}_j}{\partial \tau} - \frac{\partial^2 \bar{Z}_j}{\partial y^2} - s^2 \bar{Z}_j - FP_1(\tau, s, y) - FP_2(\tau, s, y)\right] V_j(p_j y)\, dy = 0, \quad j = 1, 2, 3,$$

it can be written that

$$\frac{d\bar{\bar{Z}}_j}{d\tau} + \left(s^2 + p_j^2\right)\bar{\bar{Z}}_j = FP_{3j}(\tau, s, p_j) + FP_{4j}(\tau, s, p_j), \quad j = 1, 2, 3, \tag{34}$$

where

$$FP_{3j}(\tau, s, p_j) = \int_{y_{2j-1}}^{y_{2j}} \left[FP_1(\tau, s, y) + FP_2(\tau, s, y)\right] V_j(p_j y)\, dy;$$

$$FP_{4j}(\tau, s, p_j) = V_j(p_j y_{2j-1}) \frac{\bar{q}u_{j+1}(\tau, s)}{b_j} + V_j(p_j y_{2j}) \frac{\alpha_{j+1}}{b_j} \bar{\theta}_{j+1}^{\,\mathrm{m}}(\tau, s). \tag{35}$$

The initial conditions for Eqs. (34) will be obtained from (15) after applying integral transformations (17) and (27) to them:

$$\bar{\bar{Z}}_j\Big|_{\tau=0} = \int_{y_{2j-1}}^{y_{2j}} y \left[\sum_{i=1}^{3} A_i \int_{x_{2i-1}}^{x_{2i}} \theta_{i0}(x, y)\, U_i(sx)\, dx\right] V_j(p_j y)\, dy, \quad j = 1, 2, 3.$$

The solutions of Eqs. (34) are

$$\bar{\bar{Z}}_j(\tau, s, p_j) = \exp\left[-\left(s^2 + p_j^2\right)\tau\right] \times$$

$$\times \left[\int_0^{\tau} \left(FP_{3j}(\tau, s, p_j) + FP_{4j}(\tau, s, p_j)\right) \exp\left[\left(s^2 + p_j^2\right)\tau\right] d\tau + \bar{\bar{Z}}_j\Big|_{\tau=0}\right], \quad j = 1, 2, 3.$$

By virtue of the orthonormalizability of the functions $U_i(sx)$, $i = 1, 2, 3$, and $V_j(p_j y)$, $i = 1, 2, 3$, the final solution of the problem has the form

$$T_1(\tau, r, z) = \sum_{p_1} \left(\sum_s \bar{\bar{Z}}_1(\tau, s, p_1)\, U_1(sx)\right) V_1(p_1 y), \quad T_2(\tau, r, z) = \sum_{p_2} \left(\sum_s \bar{\bar{Z}}_2(\tau, s, p_2)\, U_2(sx)\right) V_2(p_2 y),$$

$$T_3(\tau, r, z) = \sum_{p_3}\left(\sum_s \bar{\bar{Z}}_3(\tau, s, p_3)\, U_3(sx)\right) V_3(p_3 y) .$$

Here, summation is carried out over the positive roots s and p_j of Eqs. (21) and (32). Transition from the coordinates x, y to the coordinates r, z is performed with the aid of relations (8) and (16).

The obtained solution of the heat-conduction problem for a system of three coaxial finite cylinders with the indicated boundary conditions can be modified easily for other boundary conditions too. In the expressions presented above, in addition to the boundary conditions in initial equations (2), (3), (4), and (5) and boundary conditions (19) and (29), the equations used for determining the characteristic numbers, i.e., (21) and (32), as well as expressions (24) and (35), also undergo a change.

Consider, for example, the following boundary conditions:

on the inner cylindrical surface — the condition of the 1st kind:

$$T_1 \big|_{r=R_1} + T1(\tau, z) ;$$

on the outer cylindrical surface — the condition of the 3rd kind:

$$\lambda_3 \frac{\partial T_3}{\partial r}\bigg|_{r=R_4} + \alpha_1 \left[T_3 \big|_{r=R_4} - T_1^{\mathrm{m}}(\tau, z)\right] = 0 ;$$

on the lower end of the inner cylinder — the condition of the 1st kind:

$$T_1 \big|_{z=b} = T2(\tau, r) ;$$

on the upper end of the inner cylinder — the condition of the 3rd kind:

$$\lambda_1 \frac{\partial T_1}{\partial z}\bigg|_{z=c} + \alpha_2 \left[T_1 \big|_{z=c} - T_2^{\mathrm{m}}(\tau, r)\right] = 0 ;$$

on the lower end of the middle cylinder — the condition of the 2nd kind:

$$\lambda_2 \frac{\partial T_2}{\partial z}\bigg|_{z=b} + q_1(\tau, r) = 0 ;$$

on the upper end of the middle cylinder — the condition of the 1st kind:

$$T_2 \big|_{z=b} = T3(\tau, r) ;$$

on the lower end of the outer cylinder — the condition 3rd kind:

$$-\lambda_3 \frac{\partial T_3}{\partial z}\bigg|_{z=b} + \alpha_3 \left[T_3 \big|_{z=b} - T_3^{\mathrm{m}}(\tau, r)\right] = 0 ;$$

and on the upper end of the outer cylinder — the condition of the 2nd kind:

$$\lambda_3 \frac{\partial T_3}{\partial z}\bigg|_{z=c} - q_2(\tau, r) = 0 .$$

Then, in the system of equations (20) only the first equation will change:

$$C_1 J_0(sx_1) + C_2 Y_0(sx_1) = 0 ,$$

and the system of equations (31) will take the form

$$\begin{vmatrix} \sin(p_1 y_1) & \cos(p_1 y_1) \\ \lambda_1 p_1 \cos(p_1 y_2) + \alpha_1 \sin(p_1 y_2) & -\lambda_1 p_1 \sin(p_1 y_2) + \alpha_1 \cos(p_1 y_2) \end{vmatrix} = 0 ;$$

$$\begin{vmatrix} \cos(p_2 y_3) & -\sin(p_2 y_3) \\ \sin(p_2 y_4) & \cos(p_2 y_4) \end{vmatrix} = 0 ;$$

$$\begin{vmatrix} \lambda_3 p_3 \cos(p_3 y_5) - \alpha_3 \sin(p_3 y_5) & -\lambda_3 p_3 \sin(p_3 y_5) - \alpha_3 \cos(p_3 y_5) \\ \cos(p_3 y_6) & -\sin(p_3 y_6) \end{vmatrix} = 0 .$$

Expressions (24) and (35) will be written as

$$FP_1(\tau, s, y) = A_1 x_1 \left.\frac{dU_1}{dx}\right|_{x_1} T1(\tau, y) + A_3 x_6 U_3\Big|_{x_6} \frac{\alpha_1}{\lambda_3} T_1^{\mathrm{m}}(\tau, y) ,$$

$$FP_{41}(\tau, s, p_1) = \left.\frac{dV_1(p_1 y)}{dy}\right|_{y_1} \overline{T2}(\tau, s) + V_1(p_1 y_2) \frac{\alpha_2}{b_1} \overline{T}_2^{\mathrm{m}}(\tau, s) ,$$

$$FP_{42}(\tau, s, p_2) = V_2(p_2 y_3) \frac{\overline{q}_1(\tau, s)}{b_2} - \left.\frac{dV_2(p_2 y)}{dy}\right|_{y_4} \overline{T3}(\tau, s) ,$$

$$FP_{43}(\tau, s, p_3) = V_3(p_3 y_5) \frac{\alpha_3}{b_3} \overline{T}_3^{\mathrm{m}}(\tau, s) + V_3(p_3 y_6) \frac{\overline{q}_2(\tau, s)}{b_3} ,$$

where

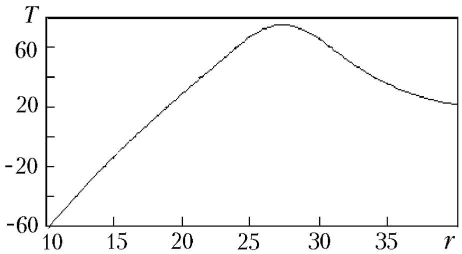
$$\overline{T1}(\tau, s) = A_1 \int_{x_1}^{x_2} x U_1(sx)\, T1(\tau, x)\, dx ; \quad \overline{T}_1^{\mathrm{m}}(\tau, s) = A_1 \int_{x_1}^{x_2} x U_1(sx)\, T_1^{\mathrm{m}}(\tau, x)\, dx ;$$

$$\overline{q}_1(\tau, s) = A_2 \int_{x_3}^{x_4} x U_2(sx)\, q_1(\tau, x)\, dx ; \quad \overline{T3}(\tau, s) = A_2 \int_{x_3}^{x_4} x U_2(sx)\, T3(\tau, x)\, dx ;$$

$$\overline{T}_3^{\mathrm{m}}(\tau, s) = A_3 \int_{x_5}^{x_6} x U_3(sx)\, T_3^{\mathrm{m}}(\tau, x)\, dx ; \quad \overline{q}_2(\tau, s) = A_3 \int_{x_5}^{x_6} x U_3(sx)\, q_2(\tau, x)\, dx .$$

The remaining relations will remain intact.

Numerical Example. Below, the boundary conditions that were presented earlier, in the beginning of the paper are used; heat release of power w_2 = 17,500 W/m^3 occurs only in the material of the middle layer of the cylinder, whereas on the outer boundaries of other adjacent layers there is heat exchange with the medium having a lower temperature (T_1^{m} = 193 K, T_2^{m} = T_3^{m} = 258 K) and the sources that absorb heat: q_1 = q_4 = –580 W/m^2; q_2 = –116 W/m^2. The remaining initial quantities are: R_1 = 0.1 m; R_2 = 0.25 m; R_3 = 0.3 m; R_4 = 0.4 m; b = 0.6 m; c = 0.8 m; q_3 = 0; w_1 = 0; w_3 = 0; a_1 = 2.2·10^{-7} m^2/sec; a_2 = 2.5·10^{-7} m^2/sec; a_3 = 2.8·10^{-7} m^2/sec; λ_1 = 0.47 W/(m·K);



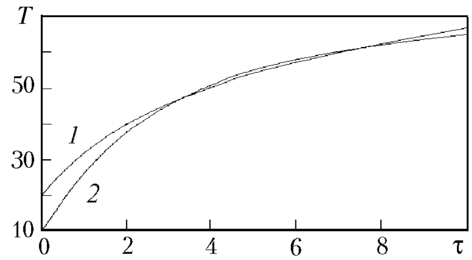


Fig. 2. Dependence of temperature on the radius of a three-layer cylinder in its middle section z = 0.7 m at τ = 15 h. r, m; T, ^{o}C.

Fig. 3. Dependence of temperature on time at the boundaries R_2 = 0.25 m (1) and R_3 = 0.3 m (2) at z = 0.7 m. τ, h; T, ^{o}C.

λ_2 = 0.7 W/(m·K); λ_3 = 0.58 W/(m·K); T_{10} = 283 K; T_{20} = 293 K; T_{30} = 303 K; $T_4^{\rm m}$ = 193 K; α_1 = 0.46 W/(m^2·K); α_2 = 0.23 W/(m^2·K); α_3 = 0.23 W/(m^2·K), and α_4 = 0.35 W/(m^2·K).

Figure 2 presents a curve demonstrating a change in the temperature over the radius of the coaxial cylinder in its middle section over the height for the time τ = 15 h from the beginning of the process. The highest temperature is observed near the center of the middle layer, which is to be expected as the energy of power w_2 = 17.5·10^3 W/m^3 is released there. In the inner and outer layers, the temperature lowers appreciably towards the boundaries, where there is a heat sink q_1 = −580 W/m^2 at the boundary $r = R_1$ and heat exchange with the medium at $T_4^{\rm m}$ = 193 K (−80^{o}C) at the boundary $r = R_4$. Moreover, it is assumed that the ends of the inner and outer cylinders are cooled by the absorbing heat sources: q_2 = −116 W/m^2, q_4 = −580 W/m^2 at $z = b$ (lower end), as well as heat exchange with the medium of lower temperature: $T_2^{\rm m}$ = $T_2^{\rm m}$ = 258 K, $T_2^{\rm m}$ = 193 K (−15^{o}C, −80^{o}C, respectively).

Figure 3 presents the curves depicting a change in the temperature with time on the outer boundary $r = R_2$ of the inner layer and on the outer boundary $r = R_3$ of the middle layer at z = 0.7 from the beginning of the process to the time value τ = 10 h. As was expected, the values of temperature at the boundaries of the middle layer ($R_2 \le r \le R_3$) are nearly the same due to its small thickness and absence of external heat release w_1 = 0, w_3 = 0 in the inner ($R_1 \le r \le R_2$) and outer ($R_3 \le r \le R_4$) layers.

NOTATION

a_i, thermal diffusivity, m^2/sec; a_{ij}, coefficient of the 1st system of equations; A_i, constant of integral transformation; b_i, thermal coefficient, W·sec$^{1/2}$/(m^2·K); b_{ij}, coefficient of the 2nd system of equations; b, c, coordinates of the lower and upper ends of the cylinder, respectively, m; C_i, D_i, arbitrary constants; f_i, coefficient of the 1st system of orthogonal functions; FP_i and FP_{ij}, right-hand sides of differential equations; g_i, coefficient of the 2nd system of orthogonal functions; J_0 and J_1, zero- and first-order Bessel functions of the first kind; p_j, root of the 2nd characteristics equation; q_i, heat-flux power on the cylinder surface, W/m^2; r, radial coordinate of the cylinder, m; R_i, radius of the cylinder, m; s_i, root of the 1st characteristic equation; T_i, temperature of the cylinder, K; T_{i0}, initial temperature of the cylinder, K; $T_i^{\rm m}$, temperature of surrounding medium, K; $T1$, $T2$, and $T3$, temperatures on the cylinder surfaces, K; $\bar{T}1$, $\bar{T}2$, and $\bar{T}3$, temperatures on the cylinder surfaces in the region of mappings; U_i and V_j, functions of integral transformations; w_i, power of internal heat release, W/m^3; x_i, replaced radial coordinate; Y_0 and Y_1, zero- and first-order Bessel functions of the second kind; z, vertical coordinate, m; $\bar{Z}_i$ and $\bar{\bar{Z}}_i$, temperature of cylinders in the region of mappings; α_i, heat-transfer coefficient, W/(m^2·K); λ_i, thermal conductivity, W/(m·K); τ, time, sec. Subscripts: i and j, numbers of functions and constants. Superscripts: m, surrounding medium; overbar and two overbars, function in the region of mappings of the first and second integral transformations, respectively.

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